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Differential algebras in non-commutative geometry

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Abstract

We discuss the differential algebras used in Connes' approach to Yang–Mills theories with spontaneous symmetry breaking. These differential algebras generated by algebras of the form functions $\otimes$ matrix are shown to be skew tensor products of differential forms with a specific matrix algebra. For that we derive a general formula for differential algebras based on tensor products of algebras. The result is used to characterize differential algebras which appear in models with one symmetry breaking scale.

Keywords: Differential algebra; Tensor product algebra; Universal differential envelope;
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1. Introduction

In spite of the great experimental success of the Standard Model of the electroweak interaction there is a general feeling that the theoretical understanding of this interaction is far from being complete. Not only are the regularities, like the appearance of the elementary particles in families, and the irregularities, like the mass-matrix, a complete mystery. Also the concept of spontaneous symmetry breaking seems to be arbitrarily introduced by hand in order to turn Yang–Mills theories into experimentally relevant models.

However, there are recent new and promising attempts to solve at least the problem related to the Higgs-mechanism and spontaneous symmetry breaking. They are more or less based on or inspired by Connes' non-commutative geometry [1]. There is one line of approach initiated by Connes himself [2], which later on was generalized to a Grand Unification Model [3,4].

There is another line of approach followed in [5,6]. It is based on superconnections in the sense of Matthai and Quillen [12]. The key-idea in these models is to extend the usual exterior differential by a matrix derivation. The physical motivation for this was

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given in [7]. The connection is then taken to be an element of a graded Lie-algebra $SU(m|n)$ which has been extended to a module over differential forms. This line of approach seems to be related to the one of Connes. However, until now it was not known to what extent they are similar and what the precise differences are. A precise comparison between the models [2,8] and [5–7] became possible only after the present construction. The results of this comparison appeared in [10].

In this article we want to investigate Connes' approach to Yang–Mills theory with spontaneous symmetry breaking. More precisely, we will discuss the differential algebra $\Omega_D\mathcal{A}$ in Connes construction, which is a noncommutative generalisation of differential forms. It is not a unique generalisation, as other noncommutative algebras have been proposed and at the time being it is not clear which one is the 'best' object. However, $\Omega_D\mathcal{A}$ is definitely an interesting object if the underlying manifold is an even dimensional compact spin manifold with euclidean signature. This algebra is a derived object, obtained from an associative algebra $\mathcal{A}$ via the universal differential enveloping algebra and a k-cycle. Therefore $\Omega_D\mathcal{A}$ is not known in general and has to be computed for each specific example. We shall give a quite detailed characterization of this object for general situations. The fact that all physical quantities like connections or curvatures are objects in the differential algebra $\Omega_D\mathcal{A}$ underlines its importance. Lower degrees of the algebra were calculated by D. Kastler and D. Testard in [9] and by A. Connes in [1].

We show that this algebra is in fact a skew tensor product of a specific differential matrix algebra with differential forms, i.e. matrix valued differential forms.

In the case of Yang–Mills theories with spontaneous symmetry breaking the algebra $\mathcal{A}$ is given as a tensor product of $\mathcal{F}$, the algebra of smooth functions, and $\mathcal{A}_\mathcal{M}$, a matrix algebra. The differential algebra $\Omega_D\mathcal{F}$ for the algebra of functions is the usual de Rham algebra [1]. $\Omega_D\mathcal{A}_\mathcal{M}$ for the matrix algebra will be easy to compute, as we shall see in Section 3. Therefore we want to make use of this fact and derive in Section 5 a general formula which relates $\Omega_D(\mathcal{A}_1 \otimes \mathcal{A}_2)$ of a product algebra to the differential algebras $\Omega_D\mathcal{A}_1$ and $\Omega_D\mathcal{A}_2$ of the factor algebras. In our case, where the tensor product of an algebra of functions and a matrix algebra is taken, the general relation becomes much simpler as we will see in Section 6. This article ends with conclusions drawn in Section 8. However, we shall first give a brief introduction to the general subject in the next section.

2. The universal differential envelope and k-cycles

We start with a brief review of the basic concepts of Connes non-commutative geometry needed to describe Yang–Mills theories with spontaneous symmetry breaking. This will allow us to fix the notation and to introduce some useful definitions. For a more comprehensive presentation of this subject we refer to [1,11,13].

Let $\mathcal{A}$ be an associative unital algebra. We can construct a bigger algebra $\Omega\mathcal{A}$ by associating to each element $A \in \mathcal{A}$ another symbolic element denoted by δA. $\Omega\mathcal{A}$ is the free algebra generated by the symbols A, δA, $A \in \mathcal{A}$ modulo relations establishing linearity and

$$\delta(AB) = \delta A\, B + A\delta B\ . \tag{2.1}$$

With the definitions

$$\delta(A_0\delta A_1\cdots\delta A_k)\ :=\ \delta A_0\,\delta A_1\cdots\delta A_k\,,\quad \delta 1\ :=\ 0\,,$$

$\Omega\mathcal{A}$ becomes a $\mathbb{N}$-graded differential algebra with the differential δ of degree $+1$, $\delta^2 = 0$. $\Omega\mathcal{A}$ is called the universal differential envelope of $\mathcal{A}$.

The next element in this formalism is a k-cycle $(\mathcal{H}, D)$ over $\mathcal{A}$, where $\mathcal{H}$ is a Hilbert space such that there is an (unit preserving) algebra homomorphism

$$\pi : \mathcal{A} \longrightarrow B(\mathcal{H})\ .$$

$B(\mathcal{H})$ denotes the algebra of bounded operators acting on $\mathcal{H}$. D is a Dirac operator such that $[D, \pi(A)]$ is bounded for all $A \in \mathcal{A}$. We can use this operator to extend π to an algebra homomorphism of $\Omega\mathcal{A}$ by defining

$$\pi(A_0\delta A_1\cdots\delta A_k) := \pi(A_0)[D,\pi(A_1)]\cdots[D,\pi(A_k)]\ .$$

However, in general $\pi(\Omega\mathcal{A})$ fails to be a differential algebra. In order to repair this, one has to divide out the two sided $\mathbb{N}$-graded differential ideal $\mathcal{J}$ given by

$$\mathcal{J} := \bigoplus \mathcal{J}^k\ ,\quad \mathcal{J}^k := \ker^k \pi + \delta \ker^{k-1}\pi\ ,\quad \ker^k \pi := \ker\pi \cap \Omega^k\mathcal{A}\ . \tag{2.2}$$

Now we are ready to define the basic object of interest, Ω_D as

$$\Omega_D := \bigoplus_{k\in\mathbb{N}} \Omega^k\mathcal{A}/\mathcal{J}^k = \bigoplus_{k\in\mathbb{N}} \pi(\Omega^k\mathcal{A})/\pi(\mathcal{J}^k)\ . \tag{2.3}$$

Ω_D is an $\mathbb{N}$-graded differential algebra, where the differential d is defined by

$$d[\pi(\omega)] := [\pi(\delta\omega)]\ ,\quad \omega \in \Omega\mathcal{A}\ .$$

As an example let $\mathcal{A}$ be the algebra of smooth functions on a compact spin-manifold, $\mathcal{H}$ the space of square-integrable spin-sections and $D = i\partial\!\!\!/$ then Ω_D is the usual de Rham-algebra [1].

So far we have only repeated the definition of Ω_D, but in this paper we shall be interested in a special kind of algebras, namely those which are built as tensor products of two algebras. Therefore we recall the notion of product k-cycles [1]. In this context the following summary will often be referred to as the setting we are interested in.

Setting: Suppose we have two k-cycles, $(\mathcal{H}_1, D_1)$ over $\mathcal{A}_1$, $(\mathcal{H}_2, D_2)$ over $\mathcal{A}_2$ and suppose there is a $\mathbb{Z}_2$-grading automorphism χ given on $\mathcal{H}_1$ such that D_1 is odd with respect to this grading. This means that there is an element $\chi \in \mathcal{B}(\mathcal{H}_1)$ with

$$\chi^2 = 1, \qquad [A,\chi] = 0, \qquad \{[D_1, A], \chi\} = 0, \qquad \text{for all } A \in \mathcal{A}_1.$$

The product k-cycle $(\mathcal{H}, D)$ over $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ is then given by

$$\mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2 , \qquad D := D_1 \otimes 1 + \chi \otimes D_2 . \tag{2.4}$$

The representations of the algebras on the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}$ shall be denoted by π_1, π_2, π, respectively.

In Section 6 we will be dealing with a product k-cycle where the first factor is derived from a compact even dimensional spin manifold. This means $\mathcal{A}_1 = \mathcal{F}$, the algebra of smooth functions, χ can be chosen as the grading automorphism of the clifford algebra, that is γ_5 for a four dimensional manifold. The second factor will be a matrix algebra represented on the Hilbert space $\mathbb{C}^n$.

3. Differential forms of associative matrix algebras

In this section we shall derive some general properties of the differential algebra generated by an associative matrix algebra $\mathcal{A}$. The k-cycle $(\mathcal{H}, D)$ over $\mathcal{A}$ is specified as

$$\mathcal{H} = \mathbb{C}^N ; \ D := [\mathcal{M}, \cdot\ ] , \ \mathcal{M} \in \mathbb{C}^{N \otimes N} . \tag{3.1}$$

Without any loss of generality we may assume that for the algebra homomorphism

$$\pi : \Omega\mathcal{A} \longrightarrow \mathbb{C}^{N\otimes N}$$

we have

$$\ker^0 \pi = \{0\} .$$

Obviously in this case $\mathcal{J}^1$ is given by

$$\mathcal{J}^1 = \ker \pi \cap \Omega^1\mathcal{A} .$$

Therefore contributions to the denominator of the second quotient in Eq. (2.3) arise only from $\pi(\mathcal{J}^k)$ with $k \geq 2$.

For quite general cases the next lemma shows that $\mathcal{J}$ is generated by $\mathcal{J}^2$ and as a consequence the differential on Ω_D is given by a supercommutator with the matrix $\mathcal{M}$ that determines the Dirac operator.

Lemma 1. *Let $\mathcal{A}$ be an associative algebra, $(\mathcal{H}, D)$ a K-cycle over $\mathcal{A}$ as in (3.1) and π the corresponding algebra homomorphism with* $\ker^0 \pi = \{0\}$. *If*

$$[\mathcal{M}^2, \pi(\mathcal{A})] \subset \pi(\mathcal{J}^2) , \tag{3.2}$$

then

i) $\pi(\mathcal{J})$ is generated by $\pi(\mathcal{J}^2)$, i.e.

$$\pi(\mathcal{J}^k) = \begin{cases} \sum_{j=0}^{k-2} \pi\big(\Omega^j\mathcal{A}\, \mathcal{J}^2\, \Omega^{k-j-2}\mathcal{A}\big) , & k \geq 2 , \\ 0 , & k = 0, 1 ; \end{cases} \tag{3.3}$$

ii) the differential d on $\Omega_D\mathcal{A}$ is given by the supercommutator

$$d[\pi(\omega^k)] = \big[[\mathcal{M}, \pi(\omega^k)]_S\big] = \big[\mathcal{M}\pi(\omega^k) - (-1)^k\pi(\omega^k)\mathcal{M}\big]$$

with $[\pi(\omega^k)] \in \Omega_D^k\mathcal{A}$ *and* $\omega^k \in \Omega^k\mathcal{A}$.

Proof. Let us consider $a_1[\mathcal{M}, a_2]\cdots[\mathcal{M}, a_k] = 0$, $a_i \in \pi(\mathcal{A})$, i.e. $\omega^k := a_1\delta a_2\cdots\delta a_k \in \ker^k\pi$. Then we have

$$\begin{aligned}\pi(\delta\omega^k) &= [\mathcal{M}, a_1]\cdots[\mathcal{M}, a_k] = -a_1\mathcal{M}[\mathcal{M}, a_2]\cdots[\mathcal{M}, a_k] \\ &= -a_1[\mathcal{M}^2, a_2]\cdots[\mathcal{M}, a_k] + a_1[\mathcal{M}, a_2]\mathcal{M}\cdots[\mathcal{M}, a_k] \\ &\;\;\vdots \\ &= -\sum_{j=2}^{k}(-1)^j a_1[\mathcal{M}, a_2]\cdots[\mathcal{M}^2, a_j]\cdots[\mathcal{M}, a_k] \\ &\quad + (-1)^k a_1[\mathcal{M}, a_2]\cdots[\mathcal{M}, a_k]\mathcal{M}\,.\end{aligned}$$

The last term in the last equation vanishes by assumption and the sum is of the form as in Eq. (3.3) which proves *i)*.

To prove the second part of the lemma we choose any $\omega^k := a_0[\mathcal{M}, a_1]\cdots[\mathcal{M}, a_k] \in \Omega^k\mathcal{A}$. Then

$$d[\pi(\omega^k)] \equiv [\pi(\delta\omega^k)] \equiv [\mathcal{M}, a_0][\mathcal{M}, a_1]\cdots[\mathcal{M}, a_k]\,.$$

On the other hand we get

$$\begin{aligned}\mathcal{M}\pi(\omega^k) &= \mathcal{M}a_0[\mathcal{M}, a_1]\cdots[\mathcal{M}, a_k] \\ &= [\mathcal{M}, a_0][\mathcal{M}, a_1]\cdots[\mathcal{M}, a_k] + a_0\mathcal{M}[\mathcal{M}, a_1]\cdots[\mathcal{M}, a_k] \\ &\;\;\vdots \\ &= [\mathcal{M}, a_0][\mathcal{M}, a_1]\cdots[\mathcal{M}, a_k] \\ &\quad - \sum_{j=1}^{k}(-1)^j a_1[\mathcal{M}, a_2]\cdots[\mathcal{M}^2, a_j]\cdots[\mathcal{M}, a_k] \\ &\quad + (-1)^k a_1[\mathcal{M}, a_2]\cdots[\mathcal{M}, a_k]\mathcal{M}\,.\end{aligned}$$

Use of condition (3.2) then results in the desired identity. □

Our next task is to find algebras for which the condition (3.2) is fulfilled. The next lemma shows that the matrix algebras which are building blocks in models discussed in [2] for the two point case meet condition (3.2). They are direct sums of algebras $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ such that the algebra homomorphism maps them into a block diagonal matrix of the form

$$\pi(\mathcal{A}) = \begin{pmatrix} \pi_1(\mathcal{A}_1) & \\ & \pi_2(\mathcal{A}_2) \end{pmatrix},$$

where π_j, $j = 1,2$ denotes the restriction of π to $\mathcal{A}_j$. The Dirac operator for those algebras is off-diagonal, which will be made more precise in the following lemma.

Lemma 2. *Let $\mathcal{A} := \mathcal{A}_1 \oplus \mathcal{A}_2$ be the direct sum of associative algebras $\mathcal{A}_i$, $i = 1,2$ with respective units 1_i. Furthermore let $(\mathcal{H}, D)$ be a K-cycle over $\mathcal{A}$ as in (3.1). If*

$$P_i \mathcal{M} P_i = 0 \,, \quad P_i := \pi(1_i) \,, \; i = 1,2 \,, \tag{3.4}$$

then

$$[\mathcal{M}^2, \pi(\mathcal{A})] \subset \pi(\mathcal{J}^2) \,.$$

Proof. For a given $A \in \mathcal{A}$ let $A_i := 1_i A 1_i$. Since

$$\omega^1 := \sum_{i=1,2} A_i \delta 1_i - 1_i \delta A_i \; \in \ker^1 \pi$$

we get $\delta\omega^1 \in \mathcal{J}^2$. On the other hand condition (3.4) implies $P_1 \mathcal{M}^2 P_2 = P_2 \mathcal{M}^2 P_1 = 0$, hence

$$\begin{aligned}\pi(\delta\omega^1) &= \sum_{i=1,2} [\mathcal{M}, \pi(A_i)][\mathcal{M}, P_i] - [\mathcal{M}, P_i][\mathcal{M}, \pi(A_i)] \\ &= \sum_{i=1,2} [\mathcal{M}^2, \pi(A_i)] \; = \; [\mathcal{M}^2, \pi(A)] \,,\end{aligned}$$

which completes the proof. □

4. Example: matrix-algebra

We now want to apply these results to a matrix algebra which is given as the direct sum of the algebras $\mathcal{A}_1 = \mathbb{C}_{n\times n}$ and $\mathcal{A}_2 = \mathbb{C}_{m\times m}$ of complex $n \times n$ resp. $m \times m$ matrices. The representation of the algebra and the Dirac operator take the form

$$\pi(\mathcal{A}) = \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix} , \quad \mathcal{M} = \begin{pmatrix} 0 & \mu^* \\ \mu & 0 \end{pmatrix} ,$$

where μ denotes an arbitrary (non-zero) complex $n \times m$ matrix. This particular setting serves as an example for what is usually referred to as the 'two point' case. Let us first consider the algebra generated by

$$\mathcal{A}_1 [\mu^* \mu, \mathcal{A}_1] \mathcal{A}_1 \,. \tag{4.1}$$

There are two possibilities. Either $\mu^*\mu \sim 1_{m\times m}$, then the commutator in (4.1) is zero and no non-trivial algebra can be generated, or $\mu^*\mu \nsim 1_{m\times m}$, then the whole algebra $\mathcal{A}_1$ is generated. There is the same situation for

$$\mathcal{A}_2 [\mu\mu^*, \mathcal{A}_2] \mathcal{A}_2$$

and therefore we may distinguish three cases:

i. $\mu^*\mu \sim 1_{m\times m}$ and $\mu\mu^* \sim 1_{n\times n}$, which is possible only for $m = n$, i.e. $\mathcal{A}_1 = \mathcal{A}_2$. In this case we have $\mathcal{J} = \{0\}$ and

$$\Omega_D^{2n}\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_1 \end{pmatrix}, \quad \Omega_D^{2n+1}\mathcal{A} = \begin{pmatrix} 0 & \mathcal{A}_1 \\ \mathcal{A}_1 & 0 \end{pmatrix}, \quad n \in \mathbb{N}. \tag{4.2}$$

The multiplication is just the ordinary matrix multiplication of $2m \times 2m$ matrices.

ii. $\mu^*\mu \not\sim 1_{m\times m}$ and $\mu\mu^* \not\sim 1_{n\times n}$. Here only $\Omega_D^1\mathcal{A}$ survives since

$$\pi(\Omega^2\mathcal{A}) = \pi(\mathcal{A}) = \mathcal{J}^2$$

and therefore

$$\pi(\Omega^k\mathcal{A}) = \mathcal{J}^k, \quad k \geq 2.$$

In this case one may view

$$\Omega_D^1\mathcal{A} = \{A \in \mathbb{C}^{m\times n}\} \oplus \{B \in \mathbb{C}^{n\times m}\}$$

as a module over $\mathcal{A}$. There is no non-trivial multiplication of elements in $\Omega_D^1\mathcal{A}$, i.e. for $\nu, \omega \in \Omega_D^1\mathcal{A}$ we have $\nu \cdot \omega = 0$.

iii. $m \leq n$, $\mu^*\mu \sim 1_{m\times m}$ and $\mu\mu^* \not\sim 1_{n\times n}$. Again we have

$$\Omega_D^1\mathcal{A} \; = \; \{A \in \mathbb{C}^{m\times n}\} \oplus \{B \in \mathbb{C}^{n\times m}\}$$

as a module over $\mathcal{A}$. However, in this case $\Omega_D^2\mathcal{A}$ is non-trivial since

$$J^2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix} \quad \Rightarrow \quad \pi(\Omega^k\mathcal{A}) = \mathcal{J}^k, \quad k \geq 3,$$

and therefore

$$\Omega_D^2\mathcal{A} \; = \; \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & 0 \end{pmatrix}$$

and all higher degrees of $\Omega_D\mathcal{A}$ are trivial. The multiplication of elements (A, B), $(A', B') \in \Omega_D^1\mathcal{A}$ is given by

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} = \begin{pmatrix} A \cdot B' & 0 \\ 0 & 0 \end{pmatrix} \in \Omega_D^2\mathcal{A},$$

where $\cdot$ denotes the usual matrix multiplication. A representation for the matrix algebra is now given by

$$\Omega_D^0\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix}, \quad \Omega_D^1\mathcal{A} = \begin{pmatrix} 0 & \eta\mathbb{C}^{m\times n} \\ \eta'\mathbb{C}^{n\times m} & 0 \end{pmatrix}, \quad \Omega_D^2\mathcal{A} = \begin{pmatrix} \eta\eta'\mathbb{C}^{n\times n} & 0 \\ 0 & 0 \end{pmatrix}.$$

The relations for the formal elements η, η' are

$$\eta\eta' \neq 0, \quad \eta'\eta = 0.$$

Although these relations seem a little awkward it is not difficult to find a representation for them. E.g.,

$$\eta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \eta' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is such a representation.

5. Ω_D of the tensor product of algebras

In this section we are going to establish a relation allowing the computation of Ω_D for a tensor product of two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ as it appears in the setting stated at the end of Section 2.

First note that the division recipe used in the definition (2.3) of Ω_D may be applied to any pair (ω, π) formed out of a graded differential algebra ω which is generated by its first two gradings and an algebraic homomorphism π. Given two such pairs, (ω_1, π_1) and (ω_2, π_2), the resulting quotient differential algebras may be identical, a simple sufficient criterion for that to happen is provided by the following lemma.

Lemma 3. *Let (ω_1, δ_1) and (ω_2, δ_2) be two graded differential algebras which are both generated by the same algebra $\mathcal{A}$ as zeroth grading and their respective differentials. For an algebra $\mathcal{B}$ and algebra homomorphisms*

$$\pi_1 : \omega_1 \to \mathcal{B}, \quad \pi_2 : \omega_2 \to \mathcal{B}$$

denote by $\mathcal{J}_{\pi_i}$ the differential ideals defined as in (2.2) and let $\Omega_{\pi_i} = \bigoplus_{k \in N} \pi_i(\omega_i^k) / \pi_i(J_{\pi_i}^k)$ be the induced differential algebras. If

$$\pi_1(a) = \pi_2(a) \ \text{ and } \ \pi_1(\delta_1 a) = \pi_2(\delta_2 a) \ \text{ for all } \ a \in \mathcal{A}$$

then

$$\Omega_{\pi_1} = \Omega_{\pi_2}.$$

Proof. As ω_1 and ω_2 are generated by elements $a_0 \delta_i a_1 \cdots \delta_i a_k$ for $i = 1, 2$, we obviously have

$$\pi_1(\omega_1) = \pi_2(\omega_2)$$

since the mappings π_i are algebra homomorphisms. Therefore the relation $a_0 \delta_1 a_1 \cdots \delta_1 a_k \in \ker^k \pi_1$ implies $a_0 \delta_2 a_1 \cdots \delta_2 a_k \in \ker^k \pi_2$ and we get

$$\pi_1(J_{\pi_1}) = \pi_2(J_{\pi_2}).$$

This establishes the identity of subalgebras of $\mathcal{B}$. □

To apply this lemma to the setting stated at the end of Section 2 consider the skew tensor product $\Omega\mathcal{A}_1 \hat{\otimes} \Omega\mathcal{A}_2$ of the differential envelopes associated with the algebras $\mathcal{A}_1$

and $\mathcal{A}_2$. It is the graded differential algebra defined as the tensored vector space together with a graded multiplication

$$(\nu_1^i \hat{\otimes} \nu_2^j)(\nu_1^k \hat{\otimes} \nu_2^l) := (-1)^{jk} \nu_1^i \nu_1^k \hat{\otimes} \nu_2^j \nu_2^l\,, \quad \nu_a^b \in \Omega_a^b \mathcal{A}$$

and a differential

$$\delta_{\hat{\otimes}}(\nu_1^i \hat{\otimes} \nu_2^j) := \delta_1 \nu_1^i \hat{\otimes} \nu_2^j + (-1)^i \nu_1^i \hat{\otimes} \delta_2 \nu_2^j\,.$$

Providing the algebraic homomorphism

$$\begin{aligned} \pi_{\hat{\otimes}} : \Omega\mathcal{A}_1 \hat{\otimes} \Omega\mathcal{A}_2 &\mapsto B(\mathcal{H}_1 \otimes \mathcal{H}_2) \\ \nu_1^i \hat{\otimes} \nu_2^j &\mapsto \pi_1(\nu_1^i)\chi^j \otimes \pi_2(\nu_2^j) \end{aligned}$$

allows the use of the last lemma with respect to the triples

$$(\Omega(\mathcal{A}_1 \otimes \mathcal{A}_2), \delta, \pi)\,, \quad (\Omega\mathcal{A}_1 \hat{\otimes} \Omega\mathcal{A}_2, \delta_{\hat{\otimes}}, \pi_{\hat{\otimes}}) \quad ,$$

resulting in the identity

$$\Omega_D(\mathcal{A}_1 \otimes \mathcal{A}_2) = \frac{\pi_{\hat{\otimes}}\big((\Omega\mathcal{A}_1 \hat{\otimes} \Omega\mathcal{A}_2)^k\big)}{\pi_{\hat{\otimes}}(\delta_{\hat{\otimes}} \ker^{k-1} \pi + \ker^k \pi)}\,. \tag{5.1}$$

To proceed further we introduce a splitting $\pi_{\hat{\otimes}} = \pi_\Sigma \circ \pi_\oplus$ by means of the algebraic homomorphisms

$$\begin{aligned} \pi_\oplus : \quad & (\Omega\mathcal{A}_1 \hat{\otimes} \Omega\mathcal{A}_2)^k && \to \bigoplus_{i+j=k} \pi_1(\Omega^i\mathcal{A}_1) \hat{\otimes} \pi_2(\Omega^j\mathcal{A}_2) \\ & \nu_1^i \hat{\otimes} \nu_2^j && \mapsto \quad \pi_1(\nu_1^i) \hat{\otimes} \pi_2(\nu_2^j) \\ \pi_\Sigma : & \bigoplus_{i+j=k} \pi_1(\Omega^i\mathcal{A}_1) \hat{\otimes} \pi_2(\Omega^j\mathcal{A}_2) && \to \quad B(\mathcal{H}_1 \otimes \mathcal{H}_2) \\ & \sum_{i+j=k} \pi(\nu_1^i) \hat{\otimes} \pi_2(\nu_2^j) && \mapsto \sum_{i+j=k} \pi(\nu_1^i)\chi^j \otimes \pi_2(\nu_2^j) \end{aligned} \tag{5.2}$$

The reason for this splitting will become clearer by calculating the quotient

$$\Omega_{\pi_\oplus} := \frac{\pi_\oplus\big((\Omega\mathcal{A}_1 \hat{\otimes} \Omega\mathcal{A}_2)^k\big)}{\pi_\oplus(J_\oplus^k)}\,, \quad J_\oplus^k = \delta_{\hat{\otimes}} \ker^{k-1} \pi_\oplus + \ker^k \pi_\oplus\,.$$

To break the denominator $\pi_\oplus(J_\oplus^k)$ appearing in the last quotient in pieces we borrow the identity

$$\ker^k \pi_\oplus = \bigoplus_{i+j=k} \ker^i \pi_1 \hat{\otimes} \Omega^j\mathcal{A}_2 + \Omega^i\mathcal{A}_1 \hat{\otimes} \ker^j \pi_2$$

from multilinear algebra and obtain the expression

$$\pi_\oplus(J_\oplus^k) = \bigoplus_{i+j=k} \pi_1(J_{\pi_1}^i) \hat{\otimes} \pi_2(\Omega^j\mathcal{A}_2) + \pi_1(\Omega^i\mathcal{A}_1) \hat{\otimes} \pi_2(J_{\pi_2}^j)\,,$$

which we now insert in the defining quotient of the algebra $\Omega_{\pi_\oplus}$:

$$
\begin{aligned}
\Omega_{\pi_\oplus} &= \frac{\bigoplus_{i+j=k} \pi_1(\Omega^i\mathcal{A}_1)\,\hat{\otimes}\,\pi_2(\Omega^j\mathcal{A}_2)}{\bigoplus_{i+j=k} \pi_1(J^i_{\pi_1})\,\hat{\otimes}\,\pi_2(\Omega^j\mathcal{A}_2) + \pi_1(\Omega^i\mathcal{A}_1)\,\hat{\otimes}\,\pi_2(J^j_{\pi_2})} \\
&= \bigoplus_{i+j=k} \frac{\pi_1(\Omega^i\mathcal{A}_1)\,\hat{\otimes}\,\pi_2(\Omega^j\mathcal{A}_2)}{\pi_1(J^i_{\pi_1})\,\hat{\otimes}\,\pi_2(\Omega^j\mathcal{A}_2) + \pi_1(\Omega^i\mathcal{A}_1)\,\hat{\otimes}\,\pi_2(J^j_{\pi_2})} \\
&= \bigoplus_{i+j=k} \frac{\pi_1(\Omega^i\mathcal{A}_1)}{\pi_1(J^i_{\pi_1})}\,\hat{\otimes}\,\frac{\pi_2(\Omega^j\mathcal{A}_2)}{\pi_2(J^j_{\pi_2})} \\
&= \bigoplus_{i+j=k} \Omega^i_D\mathcal{A}_1\,\hat{\otimes}\,\Omega^j_D\mathcal{A}_2 \\
&= \Omega_D\mathcal{A}_1\,\hat{\otimes}\,\Omega_D\mathcal{A}_2\,.
\end{aligned}
\tag{5.3}
$$

Thus we see that the differential algebra related to $\pi_\oplus$ is just the skew tensor product of the quotient differential algebras associated with $\mathcal{A}_1$ and $\mathcal{A}_2$.

But still we have to take the map π_Σ into account. To do so we have to know how the compository nature of the homomorphism π propagates into the division process used to derive the quotient algebra Ω_D.

Proposition 4. *Let (ω, d) be a $\mathbb{N}$-graded differential algebra and let $\pi = \beta\circ\alpha$ be the composition of algebraic homomorphisms into algebras ω_1 and ω_2:*

$$
\omega \xrightarrow{\alpha} \omega_1 \xrightarrow{\beta} \omega_2\,.
$$

Then we have

$$
\Omega_\pi = \bigoplus_{k\in N} \frac{\pi(\omega^k)}{\pi(J^k_\pi)} = \bigoplus_{k\in N} \frac{\beta_q(\Omega^k_\alpha)}{\beta_q(J^k_{\beta_q})}\,,\quad J^k_{\beta_q} = d_{\Omega_\alpha}\ker^{k-1}\beta_q + \ker^k\beta_q\,,
\tag{5.4}
$$

where the differential algebras $\Omega_\pi, J_\pi, \Omega_\alpha$ are defined as in lemma 3 and β_q denotes the algebraic homomorphism

$$
\beta_q:\ \bigoplus_{k\in N} \frac{\alpha(\omega^k)}{\alpha(J^k_\alpha)} \to \bigoplus_{k\in N} \frac{\beta(\alpha(\omega^k))}{\beta(\alpha(J^k_\alpha))}\,,\quad J^k_\alpha = d\ker^{k-1}\alpha + \ker^k\alpha\,.
$$

Proof. We shall go through the proof in two steps. First the vector space isomorphism is established at the level of the kth grading. Then we show that the multiplication and differential are respected thus giving an isomorphism of differential algebras.

Now $\ker\alpha \subset \ker\pi$ implies $J_\alpha \subset J_\pi$, hence we get

$$
\Omega^k_\pi = \frac{\pi(\omega^k)}{\pi(J^k_\pi)} = \frac{\pi(\omega^k)}{\pi(J^k_\alpha)}\Big/\frac{\pi(J^k_\pi)}{\pi(J^k_\alpha)} = \beta_q\left(\frac{\alpha(\omega^k)}{\alpha(J^k_\alpha)}\right)\Big/\beta_q\left(\frac{\alpha(J^k_\pi)}{\alpha(J^k_\alpha)}\right)
\tag{5.5}
$$

by definition of β_q. Next we show that

$$
\alpha(J^k_\pi)/\alpha(J^k_\alpha) = d_{\Omega_\alpha}\ker^{k-1}\beta_q + \ker^k\beta_q
\tag{5.6}
$$

and therefore we first prove the identity

$$
\ker^k\beta_q = \alpha(\ker^k\pi + J^k_\alpha)/\alpha(J^k_\alpha)\,.
\tag{5.7}
$$

Clearly '$\supseteq$' in Eq. (5.7) is given. On the other hand an element of $\ker^k \beta_q$ is represented by $\nu \in \alpha(\omega^k)$ with $\beta(\nu) \in \pi(J^k_\alpha)$. This means $\beta(\nu) = \beta \circ \alpha(j)$ for an element $j \in J^k_\alpha$ and $\nu = \alpha(\rho)$ for some $\rho \in \omega$. Now $\nu = \alpha(\rho - j) + \alpha(j)$ with $\pi(\rho - j) = 0$. Therefore $\nu \in \alpha(\ker^k \pi + J^k_\alpha)$. This gives Eq. (5.7). Eq. (5.6) is derived by:

$$\begin{aligned} d_{\Omega_\alpha} \ker^{k-1} \beta_q + \ker^k \beta_q &= d_{\Omega_\alpha} \left(\frac{\alpha(\ker^{k-1} \pi + J^{k-1}_\alpha)}{\alpha(J^{k-1}_\alpha)} \right) + \frac{\alpha(\ker^k \pi + J^k_\alpha)}{\alpha(J^k_\alpha)} \\ &= \frac{\alpha(d \ker^{k-1} \pi)}{\alpha(J^k_\alpha)} + \frac{\alpha(\ker^k \pi + d \ker^{k-1} \alpha)}{\alpha(J^k_\alpha)} \\ &= \frac{\alpha(d \ker^{k-1} \pi + \ker^k \pi)}{\alpha(J^k_\alpha)} = \frac{\alpha(J^k_\pi)}{\alpha(J^k_\alpha)} . \end{aligned}$$

On the other hand

$$\Omega^k_\alpha = \alpha(\omega^k) / \alpha(J^k_\alpha) ,$$

such that Eq. (5.5) now reads

$$\Omega^k_\pi = \beta_q(\Omega^k_\alpha) / \beta_q(J^k_{\beta_q})$$

This proves the vector space identity. Going through the proof so far it becomes clear that the isomorphism i given by Eq. (5.4) is the identity map on representatives in $\pi(\omega)$ followed by different quotient building mechanisms. The quotient in Ω_π is split up into a double quotient. Using the definition of d_{Ω_π} on Ω_π and d_{RHS} given on

$$\bigoplus_{k \in N} \beta_q(\Omega^k_\alpha) / \beta_q(J^k_{\beta_q})$$

by

$$\begin{aligned} d_{\mathrm{RHS}} [\beta_q([\alpha(\nu)]_{\alpha(J_\alpha)})]_{\beta_q(J_{\beta_q})} &= [\beta_q(d_{\Omega_\alpha} [\alpha(\nu)]_{\alpha(J_\alpha)})]_{\beta_q(J_{\beta_q})} \\ &= [\beta_q([\alpha(d\nu)]_{\alpha(J_\alpha)})]_{\beta_q(J_{\beta_q})} \end{aligned}$$

we have

$$i \circ d_{\Omega_\pi} = d_{\mathrm{RHS}} \circ i .$$

The corresponding relation holds for the multiplication defined on representatives in a similar fashion:

$$\begin{aligned} &[\nu_1]_{J_\pi} \cdot [\nu_2]_{J_\pi} = [\nu_1 \nu_2]_{J_\pi} , \\ &[\beta_q([\alpha(\nu_1)]_{\alpha(J_\alpha)})]_{\beta_q(J_{\beta_q})} \cdot [\beta_q([\alpha(\nu_2)]_{\alpha(J_\alpha)})]_{\beta_q(J_{\beta_q})} \\ &\quad = [\beta_q([\alpha(\nu_1 \nu_2)]_{\alpha(J_\alpha)})]_{\beta_q(J_{\beta_q})} , \end{aligned}$$

such that

$$i([\nu_1][\nu_2]) = i([\nu_1]) \cdot i([\nu_2]) \quad .$$

By these rules for multiplication and differential we have established an isomorphism of differential algebras. □

We now apply the proposition to the situation of a tensor product of algebras, that is to a product k-cycle given as described in the setting at the end of Section 2. So we choose

$$(\omega, d) = (\Omega\mathcal{A}_1 \hat{\otimes} \Omega\mathcal{A}_2, \delta_{\hat{\otimes}})$$

and the mappings

$$\beta = \pi_\Sigma, \quad \alpha = \pi_\oplus ,$$

with $\pi_\Sigma, \pi_\oplus$ defined by (5.2). Additional use of the identity (5.3) then leads to the following theorem.

Theorem 5.

$$\Omega_D(\mathcal{A}_1 \otimes \mathcal{A}_2) = \bigoplus_{k\in N} \frac{\pi_{\Sigma_q}\left((\Omega_D\mathcal{A}_1 \hat{\otimes} \Omega_D\mathcal{A}_2)^k\right)}{\pi_{\Sigma_q}\left(d \ker^{k-1} \pi_{\Sigma_q} + \ker^k \pi_{\Sigma_q}\right)} . \tag{5.8}$$

6. Ω_D of function ⊗ matrix algebras

For many examples of interest in particle physics the algebra has the form

$$\mathcal{A} = \mathcal{F} \otimes \mathcal{A}_\mathcal{M} \quad ,$$

that is $\mathcal{A}_1 = \mathcal{F}$ is the algebra of smooth functions on an even dimensional spin manifold and $\mathcal{A}_2 = \mathcal{A}_\mathcal{M}$ is a matrix algebra. For this special case the next lemma shows that the denominator in (5.8) vanishes. It is not necessary to go as far as the theorem of the last section since it suffices to use:

$$\Omega_D^k(\mathcal{A}_1 \otimes \mathcal{A}_2) \;=\; \frac{\pi((\Omega\mathcal{A}_1 \hat{\otimes} \Omega\mathcal{A}_2)^k)}{\pi(J_\oplus^k)} \Big/ \frac{\pi(J_\pi^k)}{\pi(J_\oplus^k)} .$$

If $\pi_1 : \mathcal{F} \to \mathcal{B}(\mathcal{H}_1)$ is a representation of smooth functions on the square-integrable spinors as described in [1], $\pi_2 : \mathcal{A}_2 \to M_n(\mathbb{C})$ an injective representation of a matrix algebra $\mathcal{A}_2$ on $\mathbb{C}^n$, we can construct a representation of $\Omega\mathcal{F} \hat{\otimes} \Omega\mathcal{A}_\mathcal{M}$ on $\mathcal{H}_1 \otimes \mathbb{C}^n$ using the derivation

$$\mathcal{D} = [D, \cdot\,] + [\gamma^5 \otimes \mathcal{M}, \cdot\,]$$

with the Dirac operator $D = i\partial_\mu\gamma^\mu$ and an $n \times n$-matrix $\mathcal{M}$. γ^5 shall denote the grading automorphism χ supposed to exist at the end of Section 2. For a four dimensional manifold this is γ^5. In the following $\mathcal{A}_2$ and $\mathcal{M}$ are assumed to satisfy condition (3.4).

Lemma 6. *With the above preliminaries*

$$\pi(J_\pi^k)/\pi(J_\oplus^k) = \{0\} .$$

Proof. During this proof we shall use the following shorthands:

$$\pi_1(\Omega^k\mathcal{F}) = \omega_1^k, \quad \pi_2(\Omega^k\mathcal{A}_\mathcal{M}) = \omega_2^k, \quad \pi_i(J_i^k) = j_i^k. \tag{6.1}$$

The important property of $\Omega(\mathcal{F})$ is for $k \geq 2$:

$$j_1^k = \omega_1^{k-2} \subseteq \omega_1^k.$$

Thus we have

$$\begin{aligned}\pi(J_\oplus^k) &= \sum_{i+j=k} j_1^i \hat{\otimes} \omega_2^j + \omega_1^i \hat{\otimes} j_2^j \\ &= \sum_{i+j=k-2} \omega_1^i \hat{\otimes} \omega_2^j + \sum_{i+j=k} \omega_1^i \hat{\otimes} j_2^j .\end{aligned} \tag{6.2}$$

We shall now choose a representative $\alpha \in \pi(J_\pi^k)$ and show that $\alpha \in \pi(J_\oplus^k)$. α can be written as $\alpha = \pi(\delta k)$ with

$$k = \bigoplus_{i+j=k-1} k_1^i \hat{\otimes} k_2^j \in \ker^{k-1}\pi \tag{6.3}$$

and

$$k_1^i = f_0^i \delta_1 f_1^i \cdots \delta_1 f_i^i, \quad k_2^j = A_0^j \delta_2 A_1^j \cdots \delta_2 A_j^j ,$$

where f are functions and $A \in \mathcal{A}_\mathcal{M}$. In Eq. (6.3) we have suppressed a further summation due to the tensor product in order to simplify notation. Then

$$\begin{aligned}\alpha &= \pi(\delta k) \\ &= \sum_{i+j=k-1} \pi_1(\delta_1 k_1^i)(\gamma^5)^j \otimes \pi_2(k_2^j) + (-1)^i \pi_1(k_1^i)(\gamma^5)^{j+1} \otimes \pi_2(\delta_2 k_2^j) \\ &= \sum_{i+j=k-1} d_D f_0^i d_D f_1^i \cdots d_D f_i^i (\gamma^5)^j \otimes A_0^j d_\mathcal{M} A_1^j \cdots d_\mathcal{M} A_j^j \\ &\quad + \sum_{i+j=k-1} (-1)^i f_0^i d_D f_1^i \cdots d_D f_i^i (\gamma^5)^{j+1} \otimes d_\mathcal{M} A_0^j d_\mathcal{M} A_1^j \cdots d_\mathcal{M} A_j^j\end{aligned} \tag{6.4}$$

with

$$d_D f = [D, f] = \pi_1(\delta_1 f), \quad d_\mathcal{M} A = [\mathcal{M}, A] = \pi_2(\delta_2 A) .$$

Since $k \in \ker^{k-1}\pi$ we also have

$$0 = \sum_{i+j=k-1} f_0^i d_D f_1^i \cdots d_D f_i^i (\gamma^5)^j \otimes A_0 d_\mathcal{M} A_1^j \cdots d_\mathcal{M} A_j^j . \tag{6.5}$$

We shall first deal with the second term of the r.h.s. of (6.4). We know from Section 3 that the differential of the matrix part can be written as a supercommutator up to elements generated by $[\mathcal{M}^2, \cdot\,]$ which are in the ideal (6.2). Thus we can rewrite this term as

$$\left[\gamma^5 \otimes M, \sum_{i+j=k-1} f_0^i d_D f_1^i \cdots d_D f_i^i (\gamma^5)^j \otimes A_0 d_{\mathcal{M}} A_1^j \cdots d_{\mathcal{M}} A_j^j\right]_S$$
$$+ \text{terms in}[\mathcal{M}^2, \cdot\,]$$

and therefore it is contained in $\pi(J^k_\oplus)$. We now turn to the first term of the r.h.s. of (6.4). Using $d_D f = [D, f]$ and Eq. (6.5) we obtain

$$\text{first term of (6.4)} = -\sum_{i+j=k-1} f_0^i D(Df_1^i) \cdots (Df_i^i)(\gamma^5)^j \otimes A_0 d_{\mathcal{M}} A_1^j \cdots d_{\mathcal{M}} A_j^j\,.$$

At least two of the γ-matrices appearing in this expression are identical, therefore this term is contained in

$$\sum_{i+j=k-2} \omega_1^i \hat{\otimes} \omega_2^j$$

and therefore in $\pi(J^k_\oplus)$ according to (6.2). This gives $\alpha \in \pi(J^k_\oplus)$. □

The theorem together with lemma 6 results in the following:

For all algebras $\mathcal{A} = \mathcal{F} \otimes \mathcal{A}_{\mathcal{M}}$ fulfilling the preliminaries of lemma 6, Ω_D is given by (using the conventions (6.1))

$$\Omega_D^k(\mathcal{F} \otimes \mathcal{A}_{\mathcal{M}}) = \frac{\omega_1^k \hat{\otimes} \omega_2^0 + \cdots + \omega_1^0 \hat{\otimes} \omega_2^k}{\omega_1^{k-2} \hat{\otimes} \omega_2^0 + \cdots + \omega_1^0 \hat{\otimes} \omega_2^{k-2} + \omega_1^{k-2} \hat{\otimes} j_2^2 + \cdots + \omega_1^0 \hat{\otimes} j_2^k}\,. \tag{6.6}$$

The differential d results from the differentials on $\Omega\mathcal{F}$ and $\Omega_D \mathcal{A}_{\mathcal{M}}$ and is given by the derivation

$$\mathcal{D} = [D, \cdot\,] + [\gamma^5 \otimes \mathcal{M}, \cdot\,]_S$$

operating on representatives of $\Omega_D(\mathcal{F} \otimes \mathcal{A}_{\mathcal{M}})$, where the supercommutator is taken with respect to the total $\mathbb{Z}_2$-grading.

Corollary 7. *$\Omega_D(\mathcal{F} \otimes \mathcal{A}_{\mathcal{M}})$ is isomorphic to the skew tensor product of the de Rham algebra $(\Lambda(M), d_c)$ and a matrix differential algebra (μ, d_μ).*

Proof. We construct a mapping that serves as a first step for the isomorphism. It is straightforward to show that

$$\phi:\ \Omega_D^k(\mathcal{F} \otimes \mathcal{A}_{\mathcal{M}}) \ \rightarrow\ \frac{\Lambda^k \hat{\otimes} \omega_2^0 + \Lambda^{k-1} \hat{\otimes} \omega_2^1 + \Lambda^{k-2} \hat{\otimes} (\omega_2^2 + \omega_2^0) + \cdots}{\Lambda^{k-2} \hat{\otimes} (j_2^2 + \omega_2^0) + \Lambda^{k-4} \hat{\otimes} (j_2^4 + \omega_2^2 + \omega_2^0) + \cdots}$$
$$[\lambda^k \hat{\otimes} \nu^0 + \lambda^{k-2} \hat{\otimes} \nu^2 + \cdots] \mapsto [[\lambda^k] \hat{\otimes} \nu^0 + [\lambda^{k-2}] \hat{\otimes} \nu^2 + \cdots] \tag{6.7}$$

is a well defined differential algebra isomorphism (here $\Omega_D(\mathcal{F} \otimes \mathcal{A}_{\mathcal{M}})$ is taken in the form of Eq. (6.6) and $[\lambda^k]$ refers to the classes $\Lambda^k = \omega_1^k / w_1^{k-2}$). Separating the expression (6.7) by the natural $\mathbb{N}$-grading of differential forms one has

$$\Omega_D^k(\mathcal{F}\otimes\mathcal{A}_{\mathcal{M}})\cong\sum_{i=0}^{k}\Lambda^i(M)\,\hat{\otimes}\,\mu^{k-1}\,, \tag{6.8}$$

where

$$\mu^j=\frac{\omega_2^j+\omega_2^{j-2}+\omega_2^{j-4}+\cdots}{j_2^j+\omega_2^{j-2}+\omega_2^{j-4}+\cdots}\,. \tag{6.9}$$

Thus we have established an algebra correspondence of $\Omega_D(\mathcal{F}\otimes\mathcal{A}_{\mathcal{M}})$ to a skew tensor product of $\Lambda(M)$ and a matrix algebra μ, given as quotient algebra obtained from $\mathcal{A}_{\mathcal{M}}$, the algebra we started with. Concerning the differential structure one obtains

$$d(\lambda\,\hat{\otimes}\,\nu)=d_c\lambda\,\hat{\otimes}\,\nu+(-1)^{\partial\lambda}\lambda\,\hat{\otimes}\,d_\mu\nu\,. \tag{6.10}$$

The differential d_μ on μ is given by the supercommutator with respect to the grading of μ

$$d_\mu([\nu])=[\mathcal{M},\nu]_S\,.\quad\square \tag{6.11}$$

7. Example: function $\otimes$ matrix-algebra

In this section we want to use the general results, developed in the previous section, to extend the example provided in Section 4. The algebra $\mathcal{A}$ is

$$\mathcal{A}=\mathcal{F}\otimes(\mathcal{A}_1\oplus\mathcal{A}_2)=\mathcal{F}\otimes\mathcal{A}_{\mathcal{M}}\,,$$

where $\mathcal{A}_1$, $\mathcal{A}_2$ denote $\mathbb{C}^{m\times m}$, resp. $\mathbb{C}^{n\times n}$ matrix algebras. The Dirac operator for $\Omega(\mathcal{A}_1\oplus\mathcal{A}_2)$ is off-diagonal as in (3.4). Thus $\Omega_D(\mathcal{A}_1\oplus\mathcal{A}_2)$ is known and we can distinguish three different cases as shown at the end of Section 4. $\Omega_D\mathcal{F}$ is the de-Rham complex [1] and the Dirac operator for the product k-cycle is

$$\mathcal{D}=i\partial\!\!\!/\otimes 1+\gamma^5\otimes\mathcal{M}\,.$$

A general result is

$$\Omega_D^0\mathcal{A}=\mathcal{A}\,,\quad \Omega_D^1\mathcal{A}=\left(\Omega_D\mathcal{F}\,\hat{\otimes}\,\Omega_D(\mathcal{A}_1\oplus\mathcal{A}_2)\right)^1$$

as one immediately infers from Eq. (6.6).

We now analyze higher degrees of $\Omega_D\mathcal{A}$ for the three different cases as in Section 4.

i. $\mu^*\mu\sim 1_{m\times m}$ and $\mu\mu^*\sim 1_{n\times n}$, i.e. $\mathcal{A}_1=\mathcal{A}_2$. Because of the isomorphism (4.2) we have:

$$\pi_1(\Omega^k\mathcal{F})\otimes\pi_2(\Omega^{l+2}\mathcal{A}_{\mathcal{M}})=\pi_1(\Omega^k\mathcal{F})\otimes\pi_2(\Omega^l\mathcal{A}_{\mathcal{M}})\,.$$

Inserting this in Eq. (6.6) and using the fact that

$$\pi_1(\mathcal{J}_{\mathcal{F}}{}^k)=\pi_1(\Omega^{k-2}\mathcal{F})\,,\quad \pi_2(\mathcal{J}^k_{\mathcal{A}_{\mathcal{M}}})=\{0\}$$

one obtains for $k=2$

$$
\begin{aligned}
\Omega_D^2\mathcal{A} &= \frac{\pi_1(\Omega^2\mathcal{F})\otimes\pi_2(\Omega^0\mathcal{A}_\mathcal{M})+\pi_1(\Omega^1\mathcal{F})\otimes\pi_2(\Omega^1\mathcal{A}_\mathcal{M})}{\pi_1(\Omega^0\mathcal{F})\otimes\pi_2(\Omega^0\mathcal{A}_\mathcal{M})} \\
&= \Lambda^2\otimes\mathcal{A}_\mathcal{M}+\Lambda^1\otimes\Omega^1\mathcal{A}_\mathcal{M}
\end{aligned}
$$

and for $k > 2$:

$$
\begin{aligned}
\Omega_D^k\mathcal{A} &= \frac{\pi_1(\Omega^k\mathcal{F})\otimes\pi_2(\Omega^0\mathcal{A}_\mathcal{M})+\pi_1(\Omega^{k-1}\mathcal{F})\otimes\pi_2(\Omega^1\mathcal{A}_\mathcal{M})}{\pi_1(\Omega^{k-2}\mathcal{F})\otimes\pi_2(\Omega^0\mathcal{A}_\mathcal{M})+\pi_1(\Omega^{k-3}\mathcal{F})\otimes\pi_2(\Omega^1\mathcal{A}_\mathcal{M})} \\
&= \Lambda^k\otimes\mathcal{A}_\mathcal{M}+\Lambda^{k-1}\otimes\Omega^1\mathcal{A}_\mathcal{M}\,.
\end{aligned}
$$

In the quotient we suppressed further terms which trivially cancel out. Λ^k denotes the space of differential forms of degree k. The degree of an element $\alpha\in\Omega_D\mathcal{A}$ is the sum of the form degree and the degree of the matrix algebra. We see that although all degrees $k\in\mathbb{N}$ of the matrix algebra $\Omega_D\mathcal{A}_\mathcal{M}$ are non-trivial, in the algebra $\Omega_D\mathcal{A}$ only the zeroth and first matrix degrees appear. However, this situation can change if we include a 'generation-space', i.e., if we allow for a bigger representation space for the algebra $\mathcal{A}_\mathcal{M}$. The homomorphism

$$
\pi_2:\mathcal{A}_\mathcal{M}\longrightarrow\mathbb{C}^{2m\times 2m}
$$

is extended to

$$
\pi_2':\mathcal{A}_\mathcal{M}\longrightarrow\mathbb{C}^{2m\times 2m}\otimes\mathbb{C}^{g\times g}\,,
$$

where $\mathbb{C}^g$ is the 'generation-space'. The new homomorphism is given as

$$
\pi_2'=\pi_2\otimes 1\,.
$$

This by itself would not yield higher matrix degrees in $\Omega_D\mathcal{A}_\mathcal{M}$. In order to get that we have to use the larger freedom in the choice of the matrix $\mathcal{M}$. We now take

$$
\mathcal{M}'=\begin{pmatrix} 0 & \mu^*\otimes G^* \\ \mu\otimes G & 0\end{pmatrix}\,.
$$

Here G denotes an arbitrary $\mathbb{C}^{g\otimes g}$-matrix. The effect of this extension is that we now can distinguish between elements $\alpha\in\pi_1(\Omega^k\mathcal{F}\otimes\Omega^p\mathcal{A}_\mathcal{M})$ and $\beta\in\pi_1(\Omega^k\mathcal{F}\otimes\Omega^q\mathcal{A}_\mathcal{M})$ for $p\neq q$ as long as the powers of G^*G resp. GG^* are linearly independent for p and q. Therefore they cannot be canceled by the denominator of Eq. (6.6). However, there is an integer $p_0\leq g$ for which the powers of the matrices become linearly dependent. In this case any element $\alpha_{p_0}\in\pi(\Omega^k\mathcal{F}\otimes\Omega^{p_0}\mathcal{A}_\mathcal{M})$ can be written as a linear combination of elements with smaller matrix degree:

$$
\alpha_{p_0}=\sum_{q=1}^{q\leq p_0/2}\alpha_{p_0-2q}\,,\quad \alpha_{p_0-2q}\in\pi(\Omega^k\mathcal{F}\otimes\Omega^{p_0-2q}\mathcal{A}_\mathcal{M})\,.
$$

As a consequence all terms with matrix degree $p\geq p_0$ in $\pi(\Omega\mathcal{A})$ are canceled by the denominator of (6.6).

These results can be summarized by defining the following representation for $\Omega_D\mathcal{A}$. The matrix part $\mathcal{A}'_{\mathcal{M}}$ of the algebra is generated by the zeroth order

$$\mathcal{A}'_0 = \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_1 \end{pmatrix}$$

and the first order

$$\mathcal{A}'_1 = \begin{pmatrix} 0 & \eta\mathcal{A}_1 \\ \eta\mathcal{A}_1 & 0 \end{pmatrix} .$$

Here we have introduced a formal element η which has the property that

$$\eta^{p_0} = 0 \, ; \quad \eta^p \neq 0 \, , \quad p < p_0$$

and it commutes with all other elements of the algebra. Thus $\mathcal{A}'_{\mathcal{M}}$ is a graded algebra with highest degree $p_0 - 1$ and an induced $\mathbb{Z}_2$ grading. The full algebra $\Omega_D\mathcal{A}$ is obtained by taking the graded tensor product of $\mathcal{A}'_{\mathcal{M}}$ and the de Rham algebra Λ

$$\Omega_D\mathcal{A} = \Lambda \hat{\otimes} \mathcal{A}'_{\mathcal{M}} \, .$$

The degree of elements in $\Omega_D\mathcal{A}$ is the sum of form degree and matrix degree. The derivation on an element $\alpha \in \Omega^k_D\mathcal{A}$ is given by

$$d\alpha = d_C\alpha + \left[\begin{pmatrix} 0 & \eta\mu^* \otimes G^* \\ \eta\mu \otimes G & 0 \end{pmatrix}, \alpha \right] ,$$

where d_C denotes the usual exterior derivative and the commutator is the graded commutator.

One now might wonder what has happened to the 'generation' space? In fact, it is not needed at the level of the algebra since it was introduced to separate the matrix degrees. This task has been taken over by the element η, which also has the nilpotency property $\eta^{p_0} = 0$. However, the 'generation'-matrix M may have a physical interpretation as a mass-matrix for fermions and one might wish to keep it in the algebra. This is of course possible and does not change any algebraic properties.

ii. $\mu^*\mu \not\sim 1_{m\times m}$ and $\mu\mu^* \not\sim 1_{n\times n}$. In this case $\Omega^1_D\mathcal{A}_{\mathcal{M}}$ is the highest matrix degree in $\Omega_D\mathcal{F} \hat{\otimes} \Omega_D\mathcal{A}_{\mathcal{M}}$ and therefore no further cancellations appear in Eq. (6.6), i.e.,

$$\ker \pi_{\Sigma_q} = \{0\} \, .$$

Thus we infer that

$$\Omega_D\mathcal{A} = \Omega_D\mathcal{F} \hat{\otimes} \Omega_D\mathcal{A}_{\mathcal{M}} = \Lambda \hat{\otimes} \Omega_D\mathcal{A}_{\mathcal{M}} \, .$$

Note that the introduction of a 'generation'-space would not change the situation.

iii. $m \leq n$, $\mu^*\mu \sim 1_{m\times m}$ and $\mu\mu^* \not\sim 1_{n\times n}$. In this case the highest matrix degree is 2 but

$$\pi(\Omega^k\mathcal{F} \otimes \Omega^2\mathcal{A}_{\mathcal{M}}) \subset \pi_2(\Omega^k\mathcal{F} \otimes \Omega^0\mathcal{A}_{\mathcal{M}}) \, .$$

Therefore the highest matrix degree in $\Omega_D\mathcal{A}$ is 1 and we can represent this algebra in the same way as in **ii.**. The extension by a 'generation'-space as in **i.** can be used to make $\pi(\Omega^k\mathcal{F}\otimes\Omega^2\mathcal{A}_{\mathcal{M}})$ and $\pi_2(\Omega^k\mathcal{F}\otimes\Omega^0\mathcal{A}_{\mathcal{M}})$ distinguishable. In this case we again have

$$\ker\pi_{\Sigma_q} = \{0\}$$

and therefore

$$\Omega_D\mathcal{A} = \Omega_D\mathcal{F}\,\hat{\otimes}\,\Omega_D\mathcal{A}_{\mathcal{M}} = \Lambda\,\hat{\otimes}\,\Omega_D\mathcal{A}_{\mathcal{M}}\ .$$

8. Conclusions

We derived a general formula which relates the differential algebra $\Omega_D(\mathcal{A}_1\otimes\mathcal{A}_2)$ of a product algebra to the differential algebras $\Omega_D\mathcal{A}_1$ and $\Omega_D\mathcal{A}_2$ of the factor algebras. This considerably simplifies the calculation of $\Omega_D(\mathcal{A}_1\otimes\mathcal{A}_2)$ once the differential algebras of the factor algebras are known.

However, in the context of Yang–Mills theories with spontaneous symmetry breaking, all relevant algebras (for Connes' model building scheme) are of the form $\mathcal{A}=\mathcal{F}\otimes\mathcal{A}_{\mathcal{M}}$ with $\mathcal{F}$, the algebra of smooth functions on space–time and $\mathcal{A}_{\mathcal{M}}$, a matrix-algebra. In this case the differential algebras of each factor algebra are known. For the algebra of functions it is the usual de Rham-algebra [1] and the differential algebras for matrix-algebras are described in Section 3. With this information it is possible to compute the full differential algebra $\Omega_D(\mathcal{F}\otimes\mathcal{A}_{\mathcal{M}})$ as we showed for the two-point case in some detail.

Since physical models, at least the bosonic part, are constructed out of objects in $\Omega_D\mathcal{A}$, which is an $\mathcal{A}$-module, the explicit knowledge of the differential algebra for a given algebra $\mathcal{A}$ allows for a very economical derivation of physical quantities like connection and curvature. This can be done in the usual way by taking an antihermitian one form as connection form and the curvature as the square of the connection. However, the construction of physically relevant models requires a more careful discussion, e.g., the imbedding of charge and iso-spin enforces a certain structure on the Higgs-sector. We shall come back to this point in a future publication.

It is now also possible to use the explicit knowledge of $\Omega_D\mathcal{A}$ to discuss the precise relation of Connes' approach to Yang–Mills theory with spontaneous symmetry breaking and the model presented in [5–7]. This latter model is based on superconnection à la Matthai, Quillen [12]. Here, the usual exterior differential is extended by a matrix differential, connections are elements of odd degree in a graded $SU(n|m)$ algebra extended to a module over differential forms. There are several features in this approach similar to the differential algebras derived in Section 6, namely the general settings in matrix valued differential forms, the matrix derivation and Cartan's derivation giving the building principles for connection and curvature. However, we also note an important difference:

the quotient building described in Section 6 does not occur there and therefore the model is not based on a differential algebra, but on an algebra with derivation.

So far, we have only discussed the bosonic sector of physical models. For the derivation of $\Omega_D\mathcal{A}$ one has to introduce a Dirac operator in order to represent the differential envelope. It is considered as a nice property of Connes' approach, that this Dirac operator can be used to write down the fermionic Lagrangian. However, if one starts with $\Omega_D\mathcal{A}$ for the construction of physical models, then there is no Dirac operator given automatically. Of course, such a Dirac operator can be derived by requiring the usual physical properties.

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